INSTITUTE FOR MATHEMATICAL RESEARCH

Universiti Putra Malaysia Mathematical Olympiad 2016
UPMO 2016

Name :

Matric No. :

Faculty :

Date : 3 April 2016

Time : 9:00 am - 12:00 noon **Duration** : 3 hours

Instruction to Candidate

1. Answer all questions.
2. Answer all questions on the answer sheets.

1. Let $z = \sqrt{y} + f(\sqrt{x} - 1)$. Determine the functions $f(x)$ and z if $z = x$ at $y = 1$.

Solution: Substituting $y = 1$ in the equation $z = \sqrt{y} + f\left(\sqrt{x} - 1\right)$ we find $x = 1 + f\left(\sqrt{x} - 1\right)$. Let us denote $\sqrt{x} - 1$ by u. Then $x = (1+u)^2$. Therefore from $x = 1 + f\left(\sqrt{x} - 1\right)$ we have,

$$\begin{aligned}(1+u)^2 &= 1 + f(u),\\ f(u) &= (1+u)^2\\ &= 1 + 2u + u^2 - 1\\ &= u^2 + 2u\end{aligned}$$

Then substituting the value of the function $f(u)$ in the original equation

$$z = \sqrt{y} + f(\sqrt{x} - 1)$$

we obtain

$$\begin{aligned}z &= \sqrt{y} + f(\sqrt{x} - 1)\\ &= \sqrt{y} + \left(\sqrt{x} - 1\right)^2 + 2\left(\sqrt{x} - 1\right)\\ &= \sqrt{y} + x - 2\sqrt{x} + 1 + 2\sqrt{x} - 2\\ &= \sqrt{y} + x - 1\end{aligned}$$

Therefore the final answer is $f(x) = x^2 + 2x$ and $z = \sqrt{y} + x - 1$.

Remark: Note that in the final answer $z = \sqrt{y} + x - 1$ the domain of z looks like the set $\{(x, y) \in \mathbb{R}^2 | y \geqslant 0\}$, but this is not the ease since the original equation involves the expression $\sqrt{x}$. Therefore, the domain of z is

$$D = \{(x, y) \in \mathbb{R}^2 | x \geqslant 0, y \geqslant 0\}.$$

2. Find the minimum value of $f(x, y, z) = x^2+y^2+\frac{z^2}{2}$, given that $x+y+z = 10$.

Solution:

$$\begin{aligned}
10 &= x + y + z \\
&= (1)x + (1)y + \frac{z}{\sqrt{2}}(\sqrt{2}) \\
&\leq \left(x^2 + y^2 + \frac{z^2}{2}\right)(1^2 + 1^2 + 2) \\
&\leq 4\left(x^2 + y^2 + \frac{z^2}{2}\right) \\
\frac{10}{4} &\leq \left(x^2 + y^2 + \frac{z^2}{2}\right) \\
\frac{5}{2} &\leq \left(x^2 + y^2 + \frac{z^2}{2}\right) \\
\therefore \left(x^2 + y^2 + \frac{z^2}{2}\right) &\geq \frac{5}{2}
\end{aligned}$$

but $f(1, 1, 1) = \frac{5}{2}$; Therefore, $\frac{5}{2}$ is the minimum value.

Now, we investigate the function to maximum.Let $x = n - 10, y = -n, z = 0$ which satisfies the equation $x + y + z = 10$, where n is positive integer. Where as,

$$f(n - 10, -n, 0) = (n - 10)^2 + n^2 \longrightarrow \infty$$

as $n \longrightarrow \infty$. Hence, maximum value of $f(x, y, z)$ do not exist.

3. Find

$$\lim_{n\to\infty} \frac{1+2^2+3^3+\cdots+n^n}{n^n}.$$

Solution:

Since

$$\begin{aligned}
1 &\leqslant \frac{1+2^2+3^3+\cdots+n^n}{n^n} \\
&\leqslant \frac{n+n^2+n^3+\cdots+n^n}{n^n} \\
&= \frac{n^{n+1}-n}{(n-1)n^n} \\
&= \frac{n^n-1}{n^n}.\frac{n}{n-1} \\
&< \frac{n}{n-1} \to 1.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \frac{1+2^2+3^3+\cdots+n^n}{n^n} = 1.$$

4. Let

$$F_n = \int_0^\infty x^n e^{-x^3} dx.$$

Show that $F_n = \dfrac{n-2}{3} F_{n-3}$ for $n \geqslant 3$.

Solution:

Let $u = x^{n-2}, du = (n-2)x^{n-3}$

$dv = x^2 e^{-x^3} dx, v = -\dfrac{e^{-x}}{3}$

$$\begin{aligned}
F_n &= \int_0^\infty x^n e^{-x^3} dx \\
&= \int_0^\infty x^{n-2} x^2 e^{-x^3} dx \\
&= \left[-\frac{x^{n-2}e^{-x^3}}{3}\right]_0^\infty + \int_0^\infty \frac{e^{-x^3}}{3}(n-2)x^{n-3} dx \\
&= \left[-\frac{x^{n-2}e^{-x^3}}{3}\right]_0^\infty + \frac{n-2}{3}\int_0^\infty x^{n-3}e^{-x^3} dx \\
&= -\left[\frac{x^{n-2}e^{-x^3}}{3}\right]_0^\infty + \frac{n-2}{3} F_{n-3}.
\end{aligned}$$

The first term on RHS:

As $x \to \infty$, $e^{-x^3} \to 0$ faster than $x^{n-2} \to \infty$.

Therefore $\left[\dfrac{x^{n-2}e^{-x^3}}{3}\right]_0^\infty \to 0$ as $x \to \infty$.

$\therefore F_n = \dfrac{n-2}{3} F_{n-3}$.

5. Calculate

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{2017}.$$

Solution:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{2} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{4} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{2017} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{(4)(504)} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

6. Find a polynomial with integer coefficients vanishing at $x = \sqrt{2} + \sqrt{3}$.

Solution:

Let us consider the polynomial $p(x) = x - \left(\sqrt{2} + \sqrt{3}\right)$. The polynomial $p(x)$ is not a polynomial with integer coefficients. Let us remove the irrationality from $p(x)$. To do this we multiply $p(x)$ by $q(x) = x^3 + \left(\sqrt{2} + \sqrt{3}\right) x^2 - \left(5 - 2\sqrt{6}\right) x - \left(\sqrt{2} - \sqrt{3}\right)$.
Indeed,

$$f(x) = p(x) \cdot q(x) = x^4 - 10x^2 + 1$$

has integer coefficients and

$$\begin{aligned} f\left(\sqrt{2} + \sqrt{3}\right) &= \left(\sqrt{2} + \sqrt{3}\right)^4 - 10\left(\sqrt{2} + \sqrt{3}\right)^2 + 1 \\ &= 0 \end{aligned}$$

The construction of $q(x)$ is as follows. Multiplying $p(x)$ by $q_1(x) = x + \left(\sqrt{2} + \sqrt{3}\right)$ we obtain

$$\begin{aligned} p(x) \cdot q_1(x) &= x^2 - \left(\sqrt{2} + \sqrt{3}\right)^2 \\ &= (x^2 - 5) - 2\sqrt{6} \end{aligned}$$

Then we multiply $p(x) \cdot q_1(x)$ by

$$q_2(x) = (x^2 - 5) + 2\sqrt{6}$$

to get

$$\begin{aligned} p(x)q_1(x)q_2(x) &= (x^2 - 5)^2 - \left(2\sqrt{6}\right)^2 \\ &= x^4 - 10x^2 + 25 - 24 \\ &= x^4 - 10x^2 + 1 \end{aligned}$$

Therefore, $q(x) = q_1(x)q_2(x)$ is the required polynomial to remove the irrationality from $p(x)$.
The answer is $f(x) = x^4 - 10x^2 + 1$.